2010 Maths

Advanced Higher

Finalised Marking Instructions

General Marking Principles

These principles describe the approach taken when marking Advanced Higher Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, can be accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values / algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated in the question paper, full credit is available for an alternative valid method. (Some likely alternatives are included but these should not be assumed to be the only acceptable ones.)

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. There is one code used M. This indicates a method mark, so in question 1(a), 1M means a method mark for the product rule.

Advanced Higher Mathematics 2010

			Marks awarded for
1. **(6)**	(a) For $f(x) = e^x \sin x^2$,		
	$f'(x) = e^x \sin x^2 + e^x\left(2x \cos x^2\right)$.	**1M**	using the Product Rule
		1,1	one for each correct term
	(b) *Method* 1		
	For $g(x) = \dfrac{x^3}{\left(1 + \tan x\right)}$,		
	$g'(x) = \dfrac{3x^2(1+\tan x) - x^3 \sec^2 x}{(1+\tan x)^2}$.	**1M**	using the Quotient Rule
		1	first term and denominator
		1	second term
	Method 2		
	$g(x) = x^3(1 + \tan x)^{-1}$	**1**	for correct rewrite
	$g'(x) =$		
	$3x^2(1+\tan x)^{-1} + x^3(-1)(1+\tan x)^{-2}\sec^2 x$	**1,1**	for accuracy
	$= \frac{x^2}{(1+\tan x)^2}\left(3 + 3\tan x - x\sec^2 x\right)$		

2. **(5)**	Let the first term be a and the common ratio be r. Then		
	$ar = -6$ and $ar^2 = 3$	**1**	{both terms needed}
	Hence		
	$r = \dfrac{ar^2}{ar} = \dfrac{3}{-6} = -\dfrac{1}{2}$.	**1**	evaluating r
	So, since $\lvert r\rvert < 1$, the sum to infinity exists.	**1**	justification
	$S = \dfrac{a}{1-r}$	**1**	correct formula
	$= \dfrac{12}{1-\left(-\frac{1}{2}\right)} = \dfrac{12}{\frac{3}{2}}$		
	$= 8$.	**1**	the sum to infinity

			Marks awarded for
3. **(7)**	(a) $t = x^4 \Rightarrow dt = 4x^3 dx$	**1**	correct differential
	$\int \frac{x^3}{1+x^8} dx = \frac{1}{4} \int \frac{4x^3}{1+(x^4)^2} dx$		
	$= \frac{1}{4} \int \frac{1}{1+t^2} dt$	**1**	correct integral in t
	$= \frac{1}{4} \tan^{-1} t + c$		
	$= \frac{1}{4} \tan^{-1} x^4 + c$	**1**	correct answer
	(b) $\int x^2 \ln x \, dx = \int (\ln x) x^2 dx$	**1M**	for using integration by parts
	$= \ln x \int x^2 dx - \int \frac{1}{x} \frac{x^3}{3} dx$	**1**	for differentiating $\ln x$
	$= \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2 dx$		
	$= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + c$	**1,1**	
4. **(4)**	The matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ gives an enlargement, scale factor 2.	**1**	correct matrix
	The matrix $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ gives a clockwise rotation of 60° about the origin.	**1**	correct matrix
	$M = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	**1**	correct order
	$= \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$	**1**	
5. **(4)**	$\binom{n+1}{3} - \binom{n}{3} = \frac{(n+1)!}{3!(n-2)!} - \frac{n!}{3!(n-3)!}$	**1**	both terms correct
	$= \frac{(n+1)!}{3!(n-2)!} - \frac{n!(n-2)}{3!(n-2)!}$		{alternative methods will appear}
	$= \frac{(n+1)! - n!(n-2)}{3!(n-2)!}$		
	$= \frac{n![(n+1)-(n-2)]}{3!(n-2)!}$	**1** **1**	correct numerator correct denominator
	$= \frac{n! \times 3}{3!(n-2)!} = \frac{n!}{2!(n-2)!}$	**1**	**1** for knowing (anywhere) $(n-2)! = (n-2) \times (n-3)!$
	$= \binom{n}{2}$		

			Marks awarded for
6. **(4)**	$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ -1 & 1 & 4 \end{vmatrix}$	**1M**	a valid approach
	$= \mathbf{i}\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}$	**1**	
	$= 9\mathbf{i} - 11\mathbf{j} + 5\mathbf{k}$	**1**	
	$\mathbf{u}.(\mathbf{v} \times \mathbf{w}) = (-2\mathbf{i} + 0\mathbf{j} + 5\mathbf{k}).(9\mathbf{i} - 11\mathbf{j} + 5\mathbf{k})$ $= -18 + 0 + 25$ $= 7.$	**1**	
7. **(6)**	$\int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)}dx$ $\frac{3x+5}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$	**1M**	
	$3x + 5 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$		
	$x = -1 \Rightarrow 2 = 2A \Rightarrow A = 1$	**1**	for first correct coefficient
	$x = -2 \Rightarrow -1 = -B \Rightarrow B = 1$ $x = -3 \Rightarrow -4 = 2C \Rightarrow C = -2$	**1**	for second correct coefficient
	Hence $\frac{3x+5}{(x+1)(x+2)(x+3)} = \frac{1}{x+1} + \frac{1}{x+2} - \frac{2}{x+3}$	**1**	for last coefficient and applying them
	$\int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)}dx = \int_1^2 \left(\frac{1}{x+1} + \frac{1}{x+2} - \frac{2}{x+3}\right)dx$		
	$= [\ln(x+1) + \ln(x+2) - 2\ln(x+3)]_1^2$ $= \ln 3 + \ln 4 - 2\ln 5 - \ln 2 - \ln 3 + 2\ln 4$	**1**	for correct integration and substitution
	$= \ln\frac{3 \times 4 \times 4^2}{5^2 \times 2 \times 3} = \ln\frac{32}{25}$	**1**	
8. **(6)**	(a) Write the odd integers as: $2n + 1$ and $2m + 1$ where n and m are integers.	**1M**	for unconnected odd integers
	Then $(2n+1)(2m+1) = 4nm + 2n + 2m + 1$ $= 2(2nm + n + m) + 1$	**1**	demonstrating clearly
	which is odd.		
	(b) Let $n = 1$, $p^1 = p$ which is given as odd.	**1**	
	Assume p^k is odd and consider p^{k+1}.	**1M**	
	$p^{k+1} = p^k \times p$	**1**	
	Since p^k is assumed to be odd and p is odd, p^{k+1} is the product of two odd integers is therefore odd.	**1**	for a valid explanation from a previous correct argument
	Thus p^{n+1} is an odd integer for all n if p is an odd integer.		

		Marks awarded for
9. **(4)** Let $f(x) = (1 + \sin^2 x)$. Then		
$f(0) = 1$	**1**	
$f'(x) = 2\sin x\cos x \Rightarrow f'(0) = 0$		
$= \sin 2x$		
$f''(x) = 2\cos 2x \Rightarrow f''(0) = 2$	**1**	
$f'''(x) = -4\sin 2x \Rightarrow f'''(0) = 0$		
$f''''(x) = -8\cos 2x \Rightarrow f''''(0) = -8$	**1**	one for each non-zero term
$f(x) = 1 + 2\frac{x^2}{2!} - 8\frac{x^4}{4!} + \ldots$	**1**	
$= 1 + x^2 - \frac{1}{3}x^4 + \ldots$		
Alternative 1		
$f(0) = 1$	**1**	
$f'(x) = 2\sin x\cos x \Rightarrow f'(0) = 0$		
$f''(x) = 2\cos^2 x - 2\sin^2 x \Rightarrow f''(0) = 2$	**1**	
$f'''(x) = 4(-\sin x)\cos x - 4\cos x\sin x \Rightarrow f'''(0) = 0$		
$f''''(x) = -8\cos^2 x + 8\sin^2 x \Rightarrow f''''(0) = -8$	**1**	
etc		
Alternative 2		
$f(x) = (1 + \sin^2 x)$		
$= 1 + \frac{1}{2} - \frac{1}{2}\cos 2x$	**1**	introducing cos2*x*
$= \frac{1}{2}(3 - \cos 2x)$		
$= \frac{1}{2}\left(3 - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \ldots\right)\right)$	**1**	expanding cos2*x*
$= \frac{1}{2}(3 - 1 + 2x - \frac{2}{3}x^4 - \ldots)$	**1**	simplifying
$= 1 + x^2 - \frac{1}{3}x^4 - \ldots$	**1**	finishing
10. **(3)** The graph is not symmetrical about the y-axis (or $f(x) \neq f(-x)$) so it is not an even function.	**1**	
The graph does not have half-turn rotational symmetry (or $f(x) \neq -f(-x)$) so it is not an odd function.	**1**	
The function is neither even nor odd.	**1**	{*apply follow through*}

			Marks awarded for
11. **(7)**	$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$$		
	$m^2 + 4m + 5 = 0$	**1**	
	$(m + 2)^2 = -1$		
	$m = -2 \pm i$	**1**	
	The general solution is		
	$y = e^{-2x}\left(A\cos x + B\sin x\right)$	**1M** **1**	appropriate CF for accuracy
	$x = 0, y = 3 \Rightarrow 3 = A$	**1**	
	$x = \frac{\pi}{2}, y = e^{-\pi} \Rightarrow e^{-\pi} = e^{-\pi}\left(3\cos\frac{\pi}{2} + B\sin\frac{\pi}{2}\right)$		
	$\Rightarrow B = 1$	**1**	
	The particular solution is:		
	$y = e^{-2x}\left(3\cos x + \sin x\right).$	**1**	
12. **(4)**	Assume $2 + x$ is rational	**1**	
	and let $2 + x = \dfrac{p}{q}$ where p, q are integers.	**1**	
	So $x = \dfrac{p}{q} - 2$		
	$= \dfrac{p - 2q}{q}$	**1**	as a single fraction
	Since $p - 2q$ and q are integers, it follows that x is rational. This is a contradiction.	**1**	
13. **(10)**	$y = t^3 - \frac{5}{2}t^2 \Rightarrow \frac{dy}{dt} = 3t^2 - 5t$	**1**	
	$x = \sqrt{t} = t^{1/2} \Rightarrow \frac{dx}{dt} = \frac{1}{2}t^{-1/2}$	**1**	
	$\Rightarrow \dfrac{dy}{dx} = \dfrac{3t^2 - 5t}{\frac{1}{2}t^{-1/2}}$	**1**	
	$= 6t^{5/2} - 10t^{3/2}$	**1**	for eliminating fractions
	$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$	**1M**	
	$= \dfrac{6 \times \frac{5}{2}t^{3/2} - 10 \times \frac{3}{2}t^{1/2}}{\frac{1}{2}t^{-1/2}}$	**1**	
	$= 30t^2 - 30t$	**1**	
	i.e. $a = 30$, $b = -30$		
	At a point of inflexion, $\frac{d^2y}{dx^2} = 0 \Rightarrow t = 0$ or 1		
	But $t > 0 \Rightarrow t = 1 \Rightarrow \frac{dy}{dx} = -4$	**1**	the value of the gradient
	and the point of contact is $\left(1, -\frac{3}{2}\right)$	**1**	
	Hence the tangent is		
	$y + \frac{3}{2} = -4\left(x - 1\right)$	**1**	
	i.e. $2y + 8x = 5$		

		Marks	Marks awarded for
14. **(10)**	$\left(\begin{array}{ccc\|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & -1 & a & 2 \end{array}\right)$	**1**	for a structured approach
	$\left(\begin{array}{ccc\|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 1 & a-2 & 0 \end{array}\right)$	**1**	
	$\left(\begin{array}{ccc\|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 2a-5 & 1 \end{array}\right)$	**1**	for triangular form
	$z = \dfrac{\mathbf{1}}{\mathbf{2a-5}}$;	**1**	one correct variable
	$2y + \dfrac{1}{2a-5} = -1 \Rightarrow 2y = \dfrac{-2a+5-1}{2a-5}$ $\Rightarrow y = \dfrac{\mathbf{2-a}}{\mathbf{2a-5}}$;		
	$x - \dfrac{2-a}{2a-5} + \dfrac{1}{2a-5} = 1$ $\Rightarrow x = \dfrac{2a-5}{2a-5} + \dfrac{1-a}{2a-5} = \dfrac{\mathbf{a-4}}{\mathbf{2a-5}}$.	**1**	for the two other variables {*other justifications for uniqueness are possible*}
	which exist when $2a - 5 \neq 0$.		
	From the third row of the final tableau, when $a = 2{\cdot}5$, there are no solutions	**1**	
	When $a = 3, x = -1, y = -1, z = 1$.	**1**	
	$AB = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$	**1**	
	From above, we have $C\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and also $A\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ which suggests $AC = I$ and this can be verified directly. Hence A is the inverse of C (or vice versa).	**2**	A candidate who obtains $AC = I$ directly may be awarded full marks.

			Marks awarded for
15.	$(x^2)^2 = 8x \Rightarrow x^4 = 8x \Rightarrow x = 0,\ 2$	**1**	values of x
	$\text{Area} = 4\int_0^2 (\sqrt{8x} - x^2)\ dx$	**1M**	$4\int_0^2$
		1	the rest
	$= 4\left[\sqrt{8}\left(\frac{2}{3}x^{3/2}\right) - \frac{1}{3}x^3\right]_0^2$	**1**	
	$= 4\left[\frac{16}{3} - \frac{8}{3}\right] = \frac{32}{3}$	**1**	
	Volume of revolution about the y-axis = $\pi\int x^2 dy$.	**1M**	
	So in this case, we need to calculate two volumes and subtract:		
	$V = \pi\left[\int_0^4 y\ dy\right] - \pi\left[\int_0^4 \frac{y^4}{64} dy\right]$	**1,1**	each term
	$= \pi\left[\frac{y^2}{2}\right]_0^4 - \pi\left[\frac{y^5}{320}\right]_0^4$	**1**	
	$= \pi\left[8 - \frac{64 \times 4^2}{320}\right]$		
	$= \frac{40 - 16}{5}\pi\left(= \frac{24\pi}{5}\right)(\approx\ 15)$	**1**	
16. **(10)**	$z^3 = r^3(\cos 3\theta + i\ \sin 3\theta)$	**1**	
	$\left(\cos\frac{2\pi}{3} + i\ \sin\frac{2\pi}{3}\right)^3 = \cos 2\pi + i\ \sin 2\pi$	**1**	necessary
	$a = 1;\ b = 0$	**1**	
	Method 1		
	$r^3(\cos 3\theta + i\ \sin 3\theta) = 8$		
	$r^3\cos 3\theta = 8$ and $r^3\sin 3\theta = 0$	**1**	
	$\Rightarrow r = 2;\ 3\theta = 0,\ 2\pi,\ 4\pi$	**1**	
	Roots are 2, $2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$, $2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$.	**1**	
	In cartesian form: 2, $(-1 + i\sqrt{3})$, $(-1 - i\sqrt{3})$	**1**	
	Method 2		
	$z^3 - 8 = 0$	**1**	
	$(z - 2)(z^2 + 2z + 4) = 0$	**1**	
	$(z - 2)\left((z + 1)^2 + (\sqrt{3})^2\right) = 0$	**1**	or by using quadratic formula
	so the roots are: 2, $(-1 + i\sqrt{3})$, $(-1 - i\sqrt{3})$	**1**	
	(a) $z_1 + z_2 + z_3 = 0$	**1**	
	(b) Since $z_1^3 = z_2^3 = z_3^3 = 8$	**1**	
	it follows that		
	$z_1^6 + z_2^6 + z_3^6 = (z_1^3)^2 + (z_2^3)^2 + (z_3^3)^2$		
	$= 3 \times 64 = 192$	**1**	

END OF SOLUTIONS